PMM U.S.S.R.,Vol.45,pp. 663-671 0021-8928/82/5 0663 $7.50/0

UDC 531.3

SECONDARY EDGE EFFECT IN THE BENDING OF THIN ELASTIC SHELLS*

L.S. SRUBSHCHIK

For the equations of strictly convex thin shells of revolution under uniform pressure and fixed clamping of the edge there is shown the existence of a unique solution corresponding to equilibrium with radial tensile forces for which the secondary edge effect phenomenon holds. There are constructed appropriate asymptotic expansions, and their foundation is given with an estimation of the remainder term. The principal terms of these expansions are represented in the form of simple computational formulas.

It is also shown that the secondary edge effect phenomenon can hold for the bending of a thin, shallow, rigidly clamped shell in the shape of an elliptical paraboloid under uniformly distributed internal pressure. Simple asymptotic formulas are written down for the appropriate solution corresponding to equilibrium with only tensile forces.

The equations of the nonlinear theory of thin elastic shells contain two essential small parameters in the highest operators: ε^2 (the relative thinness of the wall) and δ^2 (the relative loading) governed by the formulas

$$\varepsilon = h(a\gamma)^{-1},\quad \delta^2 = pa(Eh)^{-1},\quad \gamma^2 = 12(1-\sigma^2)$$

where h is the thickness and a is the characteristic dimension of the shell, p is the intensity of the transverse pressure, E is Young's modulus, and σ is the Poisson's ratio. This explains the secondary edge effect phenomenon detected in /1/, which is that as the parameters δ and $\varepsilon\delta^{-2}$ tend simultaneously to zero, the edge effect of axisymmetric bending of thin elastic shells of revolution develops within the edge effect zone of the "degenerate" problem on the equilibrium of an absolutely flexible (soft) shell.

1. Formulation of the problem.

The nonlinear differential equations of the axisymmetric deformation of rigidly clamped shells of revolution under the uniformly distributed pressure p can be written in the form /2/:

$$Av - \frac{1}{2}u^2 + \theta u = 0,\quad \varepsilon^2 Au + uv - \theta v + \frac{1}{2}q\rho^2 = 0 \tag{1.1}$$

$$A(\ ) \equiv -\rho\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\rho(\ ),\quad u = \frac{dw}{d\rho},\quad v = \frac{dF}{d\rho},\quad \theta = \frac{dz}{d\rho}$$

$$\left|\frac{v}{\rho}, \frac{u}{\rho}\right|_{\rho=0} < \infty,\quad \left[\frac{dv}{d\rho} - \frac{\sigma}{\rho}v\right]_{\rho=1} = u|_{\rho=1} = F|_{\rho=1} = 0 \tag{1.2}$$

All the quantities in (1.1) and (1.2) are dimensionless and interrleated by the dimensional formulas

$$a\{\rho, w, z\} = \{\xi, W, Z\},\quad Eh\{F, q\}\quad \left\{\frac{\Phi}{a^2},\ pa\right\},\quad \varepsilon^2 = \frac{h^2}{\gamma^2 a^2}$$

Here W is the deflection of points of the middle surface Z, Φ is the stress function, E is Young's modulus, ξ is the variable radius, and a is the radius of the reference contour. The remaining notation was introduced above. It is later assumed that the shell is strictly convex ($-m\rho \leqslant \theta \leqslant -\alpha\rho,\ 0 < \alpha < m,\ \alpha, m = \text{const}$). For a spherical shell we have $\theta = -(a/R)\rho$ (R is the radius of the sphere). The function $\theta(\rho)$ is considered sufficiently smooth. The small values of $|q|$ considered below enclose a broad class of service loads. Under external pressure (acting from the convexity of the shell) $q > 0$, while for internal pressure $q < 0$.

*Prikl.Matem.Mekhan.,45,No.5,884,894,1981

The "degenerate" problem about the equilibrium of an absolutely flexible (soft) shell later plays an important role. The appropriate equations are obtained from (1.1) and (1.2) for $\varepsilon = 0$, and are written in the form

$$Av_0 - \frac{1}{2}u_0^2 + \theta u_0 = 0, \quad (u_0 - \theta)v_0 + \frac{1}{2}q\rho^2 = 0, \quad u_0 = \frac{dw_0}{d\rho} \tag{1.3}$$
$$v_0 = \frac{dF_0}{d\rho}, \quad \left|\frac{v_0}{\rho}, \frac{u_0}{\rho}\right|_{\rho=0} < \infty,$$
$$\left[\frac{dv_0}{d\rho} - \frac{\sigma}{\rho}v_0\right]_{\rho=1} = w_0(1) = F_0(1) = 0$$

The problem (1.3) has many solutions /3,4/. However, the so-called positive solutions ($v_0 \geqslant 0$). which correspond to equilibria without radial compressive stresses /5/, have meaning in mechanics. It is known /3/ that a positive solution exists and is unique, and in its neighborhood there is a solution of the problem (1.1) and (1.2) as $\varepsilon \to 0$ for which $v \geqslant 0$ (such solutions are called membrane solutions in /3/), and the following asymptotic expansions are valid

$$v \sim \sum_{s=0}^{n+1} \varepsilon^s [v_s(\rho) + \varepsilon h_{s+1}(t)], \quad h_1 = 0 \tag{1.4}$$
$$u \sim \sum_{s=0}^{n} \varepsilon^s [u_s(\rho) + g_s(t)], \quad t = \frac{1-\rho}{\varepsilon}$$

The functions u_s, v_s are obtained by direct expansion of the solution in powers of ε and are determined from the boundary value problems

$$Av_s - \frac{1}{2}\sum_{k+j=s} u_k u_j + \theta u_s = 0, \quad \sum_{k+j=s} u_k v_j - \theta v_s + Au_{s-2} = 0 \tag{1.5}$$
$$\left|\frac{v_s}{\rho}\right|_{\rho=0} < \infty, \quad \left[\frac{dv_s}{d\rho} - \frac{\sigma}{\rho}v_s\right]_{\rho=1} = \left[\frac{dh_{s+1}}{dt} + \sigma h_s\right]_{t=0}, \quad s \geqslant 1$$

The boundary-layer functions h_s, g_s are determined from second order differential equations with constant coefficients and are written down in explicit form if the functions v_i, u_i have already been found for $i \leqslant s$ (see (3.6)—(3.9) in /3/).

Furthermore, for $|q| \ll 1$ simple asymptotic formulas are constructed by the asymptotic boundary-layer method /6,7/ for the positive solution of the problem (1.3) correspondong to the solutions of the problems (1.5), and therefore, for the membrane solution of the problem (1.1), (1.2). Let us note that the principal terms of the asymptotic for the problem (1.3) had been obtained formally earlier /1,8—10/ for $q < 0, q \to 0$.

For $q > 0$ the shell is first "inverted", afterwards the applied load causes an increase in shell convexity, and as will be shown below, takes on an equilibrium mode close to the surface $(-Z)$, which is a mirror image of the bulging initial shell surface relative to the plane of the reference contour. This varifies the A.V. Pogorelov hypothesis about the existence of such equilibrium modes (see the "principle" 4 in /11/). Let us note that for $|q| \to 0$ the foundation of the asymptotic of the membrane solution as $\varepsilon \to 0$ generally drops out since the constant in the **a priori** estimate (4.9) in /3/ grows without limit. However, analogous considerations for the requirement of a simultaneous tendency of the parameters δ and $\varepsilon\delta^{-2}$ to zero permit carrying the proof of the existence of the membrane solution and the foundation of its asymptotic expansions in /3/ over to this limit case as well.

2. Asymptotic of the positive solution for an absolutely flexible shells under small loads. As $|q| \to 0$ the problem (1.3) is a singular perturbed problem with a small parameter q in the highest operator. In fact, by setting $q = \delta^2$, $v_0 = f\delta^2$, $u_0 = y$ we obtain from (1.3) for $q > 0$

$$\delta^2 Af - \frac{1}{2}y^2 + \theta y = 0, \quad (y - \theta)f + \frac{1}{2}\rho^2 = 0 \tag{2.1}$$
$$f = \frac{dF_0}{d\rho}, \quad y = \frac{dw_0}{d\rho}, \quad |f\rho^{-1}|_{\rho=0} < \infty$$
$$\left[\frac{df}{d\rho} - \frac{\sigma}{\rho}f\right]_{\rho=1} = 0, \quad w_0(1) = F_0(1) = 0, \quad 0 < \sigma < \frac{1}{2}$$

Asymptotic expansions of the positive solution of the problem (2.1) are constructed as $\delta \to 0$ in the form

$$f \sim f_\delta^n = \sum_{k=0}^{n+1} \delta^k [f_k(\rho) + \psi_k(\tau)], \quad y \sim y_\delta^n = \sum_{k=0}^{n+1} \delta^k [y_k(\rho) + \varphi_k(\tau)] \tag{2.2}$$

where $\tau = (1 - \rho)/\delta$. The functions f_k, y_k are obtained by using the first iteration process (6,7) for the direct expansion of the solution in integer powers of the parameter δ. Equating coefficients of $\delta^0, \delta^1, \ldots, \delta^n$ to zero, we deduce successively from (2.1)

$$y_0 = 2\theta,\ f_0 = -\frac{1}{2}\rho^2\theta^{-1},\ y_1 = f_1 = 0,\ y_k = \theta^{-1}\Big[Af_{k-2} - \frac{1}{2}\sum_{i=1}^{k-1} y_i y_{k-i}\Big],\ f_k = -\theta^{-1}\Big[\sum_{i=1}^{k-1} y_i f_{k-i} + y_k f_0\Big]\ (k \geqslant 2) \quad (2.3)$$

This same iteration process yields still another family y_k, f_k ($y_0 = 0,\ f_0 = {}^1/_2\rho^2\theta^{-1}, \ldots$) which is not examined later since f_0 does not satisfy the positivity condition.

Functions ψ_k, φ_k of the boundary-layer type are obtained by using the second iteration process /6,7/. Namely, we substitute (2.2) into (2.1), take account of (2.3), expand the functions f_k, y_k, θ in Taylor series at the point $\rho = 1$, set $\rho = 1 - \delta\tau$ and equate the coefficients of $\delta^0, \delta^1, \ldots, \delta^n$ successively. We obtain the system

$$\psi_0'' + \frac{1}{2}\varphi_0^2 + \theta_0\varphi_0 = 0,\quad f_{00}\varphi_0 + \psi_0\varphi_0 + \theta_0\psi_0 = 0,\quad (\ )' = d(\ )/d\tau \quad (2.4)$$
$$\psi_0'|_{\tau=0} = 0,\quad \psi_0|_{\tau=\delta^{-1}} = 0,\quad \theta_0 = \theta(1),\quad f_{00} = f_0(1),\quad \tau = (1-\rho)/\delta$$

to determine ψ_0, φ_0.

Evidently $\varphi_0 = \psi_0 = 0$. Taking this fact into account, we deduce for the determination of ψ_i, φ_i ($i \geqslant 1$)

$$\psi_i'' - \alpha_0^2\psi_i = R_{i1} - \theta_0 R_{i2} f_{00}^{-1},\quad \varphi_i = f_{00}^{-1}(R_{i2} - \theta_0\psi_i) \quad (2.5)$$

$$\alpha_0^2 = \frac{\theta_0^3}{f_{00}},\quad R_{i1} = \tau\psi_{i-1}'' + \psi_{i-1}' + \sum_{j+m+2=i}\tau^j\psi_m - \sum_{m+r+p=i}\tau^r y_{mr}\varphi_p - \frac{1}{2}\sum_{m+p=i}\varphi_m\varphi_p + \sum_{m+p=i}\tau^m\theta_m\varphi_p$$

$$R_{i2} = \sum_{j=1}^{i-1}\varphi_j\psi_{i-j} + \sum_{m+r+p=i}\tau^r y_{mr}\psi_p - \sum_{m+r+p=i}\tau^r f_m\varphi_p - \sum_{m+p=i}\theta_m\tau^m\psi_p,\quad \psi_i'(0) = -\sigma\psi_{i-1}(0) + \Big[\frac{df_{i-1}}{d\rho} - \sigma f_{i-1}\Big]_{\rho=1}$$
$$\{\theta_l, y_{ml}, f_{ml}\} = \frac{(-1)^l}{l!}\frac{d^l}{d\rho^l}\{\theta, y_m, f_m\}_{\rho=1},\quad \psi_i\Big(\frac{1}{\delta}\Big) = 0,\quad p \neq i$$

Here R_{i1}, R_{i2} are known functions if $y_k, f_k, \varphi_k, \psi_k$, have already been found for $0 \leqslant k \leqslant i-1$. In particular, we have $R_{11} = R_{12} = 0$, and we find from (2.5)

$$\psi_1 = -f_{00}\theta_0^{-1}\varphi_1 = C\exp(-\alpha_0\tau),\quad \alpha_0^2 = -2\theta_0^3 > 0 \quad (2.6)$$
$$C = 2^{-1,5}(-\theta_0)^{-3,5}[(2-\sigma)\theta_0 + \theta_1],\quad \theta_1 = -\frac{d\theta}{d\rho}\Big|_{\rho=1}$$

Evidently ψ_i, φ_i are zero-order boundary-layer functions.

3. Foundation for the asymptotic expansions (2.2). We use the method developed in /3,12–14/ for the foundation of the asymptotic expansions (2.1). We introduce the notation /14/

$$f^* = f_\delta^n + \sum_{i=1}^{n+1}\delta^i\eta_i + \delta^{n+1}\gamma_2,\quad y^* = y_\delta^n + \sum_{i=1}^{n+1}\delta^i\xi_i \quad (3.1)$$

where η_i, ξ_i are of an exponential order of smallness in δ infinitely differentiable monotonic functions, where $\eta_i(\rho) = -\psi_i(\delta^{-1})$, $\xi_i(\rho) = -\varphi_i(\delta^{-1})$ for $0 \leqslant \rho \leqslant 0{,}1$ and $\eta_i(\rho) = \xi_i(\rho) = 0$ for $0.2 \leqslant \rho \leqslant 1$. Evidently the functions f^*, y^* satisfy all the boundary conditions in (2.1) and the following estimates hold

$$\max\left|\frac{d^\varkappa z}{d\rho^\varkappa}\right| < c_0\delta^{n+1},\quad \varkappa = 0, 1, 2 \quad (3.2)$$

where $z = f_\delta^n - f^*$ or $z = y_\delta^n - y^*$. Here, as everywhere in Sect.2, the c_i are certain positive constants independent of ρ and δ; the maximum is taken everywhere for $0 \leqslant \rho \leqslant 1$.

Let us introduce Banach spaces of the functions: 1) the space X of the vector functions $\mathbf{U} = \{F_0, w_0\}$ with the components $F_0 \in C_4[0,1]$ and $w_0 \in C_2[0,1]$ which satisfy the boundary conditions in (2.1) and the norm defined by the equality $\|\mathbf{U}\|_X = |F_0|_4 + |w_0|_2$; 2) the space Y of vector functions $\mathbf{b} = \{b_1, b_2\}$ with the components $b_1, b_2 \in C_0[0,1]$ and the norm defined by equality $\|b\|_Y = |b_1|_0 + |b_2|_0$. The spaces $C_k[0,1]$ are formed by functions defined in the segment $[0,1]$ which have continuous derivatives to order k inclusive. The norm $|\cdot|_k$ in $C_k[0,1]$ is defined in the usual manner.

Let us examine the problem (2.1) as an operator equation $\mathbf{R}(\mathbf{U}) = 0$, where $\mathbf{U} = \{F_0, w_0\}$ is a solution, and the operator $\mathbf{R}$ is defined by the left side of the system (2.1) and acts from the space X into the space Y.

According to /3,12–14/, to give a foundation to the asymptotic, it is necessary to prove that the inequality

$$\|\mathbf{R}(\mathbf{U}^*)\|_Y \|[\mathbf{R}'_{\mathbf{U}^*}]^{-1}\|^2_{(Y\to X)} \|\mathbf{R}''\|_{(X\to(X\to Y))} < \frac{1}{2} \tag{3.3}$$

is satisfied as $\delta \to 0$, where $\mathbf{U}^* = \{F_0^*, w_0^*\}$ is the constructed asymptotic expansion of (3.1), $\mathbf{R}'_{\mathbf{U}^*}$ is the Fréchet derivative on the element $\mathbf{U}^*$, and $\mathbf{R}''$ is the second derivative of the operator $\mathbf{R}$.

Lemma 3.1. Let $\delta \to 0$ and $-m\rho \leqslant \theta \leqslant -\alpha\rho$, $0 < \alpha < m$, α, m = const. Then the following estimates hold

$$f^* \geqslant \frac{1}{4\alpha}\rho, \quad \max|\rho^{-1}(y^* - \theta)| \leqslant m + 1 = m_0 \tag{3.4}$$

Proof. Taking account of the inequalities

$$\max\left|\frac{\psi_i + \eta_i}{\rho}\right| \leqslant \max\left|\frac{d\psi_i}{d\rho}\right|, \quad \max\left|\frac{\varphi_i + \xi_i}{\rho}\right| \leqslant \max\left|\frac{d\varphi_i}{d\rho}\right|$$

we rewrite f^* and y^* in the form

$$f^* = -\frac{1}{2}\rho^2\theta^{-1} + \delta(\psi_1 + \eta_1) + O(\rho\delta^2), \quad y^* = 2\theta + \delta(\varphi_1 + \xi_1) + O(\rho\delta^2)$$

from which we deduce (3.4) as $\delta \to 0$ by virtue of the conditions of the lemma.

Lemma 3.2. Let the conditions of Lemma 3.1 be satisfied. Then the following estimates hold

$$\|\mathbf{R}(\mathbf{U}^*)\|_Y \leqslant c_1\delta^{n+1}, \quad \|\mathbf{R}''\| < c_3 \tag{3.5}$$

The proof is analogous to the proof of Lemmas 2.2 and 2.3 in /14/ and the estimate (4.17) in /3/.

Theorem 3.1. Let the conditions of Lemma 3.1 be satisfied. Then the following estimate holds

$$\|[\mathbf{R}'_{\mathbf{U}^*}]^{-1}\|_{(Y\to X)} \leqslant c_2\delta^{-6} \tag{3.6}$$

Proof. We consider the system of equations with boundary conditions

$$\mathbf{R}_{\mathbf{U}^*}'(\mathbf{U}) = \mathbf{b}, \quad \mathbf{b} = (b_1, b_2), \quad \mathbf{U} = (F, w) \tag{3.7}$$

$$\mathbf{R}'_{\mathbf{U}^*} = \left\{\delta^2\nabla^4 F + \frac{1}{\rho}\frac{d}{d\rho}[u(y^* - \theta)], \quad -\frac{1}{\rho}\frac{d}{d\rho}[f^* u + v(y^* - \theta)]\right\}$$

$$\nabla^2 = \frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}, \quad v = \frac{dF}{d\rho}, \quad u = \frac{dw}{d\rho}$$

$$\left|\frac{v}{\rho}, \frac{u}{\rho}\right|_{\rho=0} < \infty, \quad \left[\frac{dv}{d\rho} - \frac{\sigma}{\rho}v\right]_{\rho=1} = w(1) = F(1) = 0$$

Multiplying the first equation in (3.7) by ρF, the second by ρw, adding the expressions obtained and integrating between 0 and 1, we deduce, by taking account of the boundary conditions

$$\delta^2\int_0^1\left[\rho\left(\frac{dv}{d\rho}\right)^2 + \frac{v^2}{\rho}\right]d\rho - \delta^2\sigma v^2(1) + \int_0^1 f^* u^2\,d\rho = \int_0^1 (b_1F + b_2w)\rho\,d\rho \tag{3.8}$$

Furthermore, taking into account that $F(1) = w(1) = 0$, we find the simple inequalities

$$\int_0^1 \rho F^2\,d\rho \leqslant \frac{1}{8}\int_0^1 \frac{v^2}{\rho}\,d\rho, \quad \int_0^1 \rho w^2\,d\rho \leqslant \frac{1}{2}\int_0^1 \rho\left(\frac{dw}{d\rho}\right)^2 d\rho \tag{3.9}$$

Now, by using (3.4), (3.9) and the Cauchy–Buniakovskii inequality, we successively obtain from (3.8)

$$\delta^2(1-\sigma)\|v\|_H^2 + \frac{1}{4\alpha}\|u\|_\rho^2 \leqslant \|b_1\|_\rho\|F\|_\rho + \|b_2\|_\rho\|w\|_\rho \leqslant 2^{-1/2}(\|b_1\|_\rho\|v\|_H + \|b_2\|_\rho\|u\|_\rho) \tag{3.10}$$

$$\|v\|_H^2 = \int_0^1\left[\rho\left(\frac{dv}{d\rho}\right)^2 + \frac{v^2}{\rho}\right]d\rho, \quad \|v\|_\rho^2 = \int_0^1 \rho v^2\,d\rho$$

The following estimates hence result directly

$$\| v \|_H + \| u \|_\rho \leqslant c_4 \delta^{-2} (\| b_1 \|_\rho + \| b_2 \|_\rho) \tag{3.11}$$

$$\max | v | \leqslant c_4 \delta^{-2} \| \mathbf{b} \|_Y, \quad c_4 = \sqrt{2} (1 - \sigma)^{-1}, \quad \delta^2 < [4\alpha (1 - \sigma)]^{-1}$$

Integrating (3.7) between 0 and ρ, we have

$$\delta^2 A v - u (y^* - \theta) = B_1, \quad f^* u + v (y^* - \theta) = B_2 \tag{3.12}$$

$$B_1 = -\int_0^\rho t b_1 \, dt, \qquad B_2 = -\int_0^\rho t b_2 \, dt, \qquad \left| \frac{v}{\rho}, \frac{u}{\rho} \right|_{\rho=0} < \infty, \qquad \left[\frac{dv}{d\rho} - \sigma v \right]_{\rho=1} = 0$$

Applying (3.4) and (3.11), we deduce from the second equation

$$\max | u | \leqslant c_5 \delta^{-2} \| b \|_Y, \quad c_5 = 4\alpha (c_4 m_0 + 1) \tag{3.13}$$

Let us go from the first equation in (3.12) to the equivalent integral equation

$$\delta^2 v = -\frac{1}{\rho} \Phi (\rho, 1) + \rho \frac{1 + \sigma}{1 - \sigma} \Phi (1, 1) \tag{3.14}$$

$$\Phi (\rho, t) = \int_0^\rho \eta \, d\eta \int_\eta^t B_3 \xi^{-1} d\xi, \qquad B_3 = B_1 + u (y^* - \theta)$$

Differentiating (3.14) twice with respect to ρ, we obtain

$$\delta^2 \frac{dv}{d\rho} = \int_\rho^1 B_3 \xi^{-1} d\xi - \frac{1}{\rho^2} \Phi (\rho, 1) + \frac{1 + \sigma}{1 - \sigma} \Phi (1, 1) \tag{3.15}$$

$$\delta^2 \frac{d^2 v}{d\rho^2} = - B_3 \rho^{-1} + 2\rho^{-3} \Phi (\rho, \rho)$$

From the first equation in (3.15) we find the estimate

$$\max \left| \frac{dv}{d\rho} \right| \leqslant c_6 \delta^{-4} \| \mathbf{b} \|_Y, \qquad c_6 = \frac{3}{2} (1 + c_5 m_0) \tag{3.16}$$

Furthermore, we divide the second equation in (3.12) by f^* and differentiate the expression obtained with respect to ρ. Then applying (3.4), (3.9), (3.14) and (3.16), as well as the inequalities

$$\max \left| \frac{df^*}{d\rho} \right| + \max \left| \frac{d}{d\rho} (\theta - y^*) \right| \leqslant c_7, \qquad m_0 \leqslant c_7$$

we derive the estimates

$$\max \left| \frac{du}{d\rho} \right| \leqslant c_8 \delta^{-4} \| \mathbf{b} \|_Y, \qquad c_8 = 4\alpha c_7 [1 + c_6 + c_7 + 4\alpha c_5 c_7] \tag{3.17}$$

By using (3.17) and the triangle inequality, we have from the second equation in (3.15)

$$\max \left| \rho^{-1} \frac{d^2 v}{d\rho^2} \right| \leqslant c_9 \delta^{-6} \| \mathbf{b} \|_Y, \qquad c_9 = \frac{3}{2} m_0 c_8 + 1 \tag{3.18}$$

Finally, we obtain the estimate

$$\max \left| \frac{d^3 v}{d\rho^3} \right| \leqslant c_{10} \delta^{-6} \| \mathbf{b} \|_Y, \qquad c_{10} = c_9 + 2 c_7 c_8 + c_{11} + 1 \tag{3.19}$$

where $c_{11} = 1 + m_0 c_5$. For this we find from (3.15)

$$\delta^2 \frac{1}{\rho} \frac{d}{d\rho} \left(\frac{v}{\rho} \right) = - 2\rho^{-4} \Phi (\rho, \rho)$$

By using (3.4) and (3.18) we hence deduce

$$\max \left| \frac{1}{\rho} \frac{d}{d\rho} \left(\frac{v}{\rho} \right) \right| \leqslant c_{11} \delta^{-6} \| \mathbf{b} \|_Y \tag{3.20}$$

We later differentiate the first equation in (3.12) with respect to ρ, and by using the triangle inequalties as well as the estimates (3.4), (3.17), (3.18), (3.20), we arrive at (3.19). The estimate (3.6) now results from (3.11), (3.13), (3.16)—(3.20), where $c_2 = 2c_6 + c_9 + c_{10}$.

Theorem 3.2. Let the conditions of Lemma 3.1 be satisfied. Then there exists a q_1 such that the problem (2.1) has a unique positive solution for $0 < q < q_1$ for which the asymptotic expansions (2.2) are valid and the following estimates hold

$$\max|f - f_\delta^k| + \max|y - y_\delta^k| \leqslant c_{12}\delta^{k+2}, \quad q = \delta^2 \tag{3.21}$$
$$\max\left|\frac{d}{d\rho}(f - f_\delta^k)\right| + \max\left|\frac{d}{d\rho}(y - y_\delta^k)\right| \leqslant c_{13}\delta^{k+1}, \quad k \geqslant 0$$

The proof is carried out by using Theorem 3.2 in /13/.

The inequality (3.3) is satisfied if q in the expansions (3.1) is sufficiently small $(0 < q < q_1)$ and $n > 11$. By using (2.2), (3.1), (3.2) we find from (3.20) in /13/

$$\| U - U^* \|_Y \leqslant c_{12}\delta^{n-3} \; (n > 3) \tag{3.22}$$

We hence arrive at (3.21) by using the triangle inequality and embedding theorems.

4. Construction of the principal terms of the asymptotic expansions (1.4) as $q \to 0$. The boundary value problems (1.5) are a singularly perturbed system of linear differential equations with a small parameter $|q|$ in the highest operator and with variable coefficients containing boundary-layer-type functions. In fact, by using the change of variables $q = \delta^2$:

$$v_s = \delta^{2-2s}\nu_s, \quad u_s = \delta^{-2s}\omega_s, \quad \nu_0 = f, \; \omega_0 = y \tag{4.1}$$
$$\frac{d^\varkappa h_{s+2}}{dt^\varkappa} = \delta^{\varkappa-2s-2}\frac{d^\varkappa h_{s+2}^\circ}{dt^\varkappa}, \quad \frac{d^\varkappa g_s}{dt^\varkappa} = \delta^{\varkappa-2s}\frac{d^\varkappa g_s^\circ}{dt^\varkappa}, \quad q > 0$$

we obtain from (1.5) for $\varkappa = 0, 1, 2$

$$\delta^2 A\nu_s - \frac{1}{2}\sum_{k+j=s}\omega_k\omega_j + \theta\omega_s = 0, \quad s \geqslant 1 \tag{4.2}$$
$$\sum_{k+j=s}\nu_k\omega_j - \theta\nu_s + \delta^4 A\omega_{s-2} = 0, \quad \omega_{-1} = 0$$
$$\left|\frac{\nu_s}{\rho}\right|_{\rho=0} < \infty, \quad \left[\frac{d\nu_s}{d\rho} - \sigma\nu_s\right]_{\rho=1} = \delta^{-1}\left[\frac{dh_{s+1}^0}{dt} + \delta\sigma h_s^0\right]_{t=0}$$

The functions ν_i, ω_i are found successively, and are considered known for $i < s$, where ν_0, ω_0 are defined by (2.2). The asymptotic expansions of the boundary value problems (4.2) are constructed as $\delta \to 0$ in the form

$$\nu_s \sim \sum_{k=0}^{n_s}\delta^k[\nu_{sk}(\rho) + \Pi_k\nu_s(\tau)] \tag{4.3}$$
$$\omega_s \sim \sum_{k=0}^{n_s}\delta^k[\omega_{sk}(\rho) + \Pi_k\omega_s(\tau)], \quad \tau = (1-\rho)/\delta$$

The functions ν_{sk}, ω_{sk} are obtained by direct expansion of the solution (ν_s, ω_s) in powers of δ and are determined from simple algebraic equations

$$\theta\omega_{s0} - \sum_{k+j=s}\omega_{k0}\omega_{j0} = 0, \quad \sum_{k+j=s}\nu_{k0}\omega_{j0} - \theta\nu_{s0} = 0 \tag{4.4}$$
$$A\nu_{s,l-2} - \frac{1}{2}\sum_{k+j=s}\sum_{m+i=l}\omega_{km}\omega_{ji} + \theta\omega_{sl} = 0$$
$$\sum_{k+j=s}\sum_{m+i=l}\omega_{km}\nu_{ji} - \theta\nu_{sl} + A\omega_{s-2,k-4} = 0, \quad s, l \geqslant 1$$
$$q = -1, -2; k, \; m, j, i \geqslant 0; \quad \omega_{k,p} = \nu_{s,q} = 0, p = -1, -2, -3$$

The boundary layer functions $\Pi_i\nu_s, \Pi_i\omega_s$ are constructed by using the second iteration process /7/. Equations of the form (2.5) with right sides dependent on $\Pi_j\nu_s, \Pi_j\omega_s$ for $j < i$, on $\Pi_k\nu_m, \Pi_k\omega_m$ for $m < s$, and also on the coefficients of the Taylor series expansions of the functions v_{js}, ω_{js} for $\rho = 1$, etc., are obtained for their determination.

Limiting ourselves to the principal terms of the asymptotic expansions (2.2), (4.2), by using (2.3)—(2.6) as well as (3.6)—(3.9) from /3/, we have for the membrane solution from (1.4) for $n = 1$ in the case $q > 0$ as the parameters δ and $\varepsilon\delta^{-2}$ tend simultaneously to zero

$$\begin{aligned}
&v \sim v_{\varepsilon,\delta} = v_0(\rho, \tau) + \varepsilon v_1(\rho, \tau) + \varepsilon^2 h_2(t), \quad t = (1-\rho)/\varepsilon \\
&u \sim u_{\varepsilon,\delta} = u_0(\rho, \tau) + g_0(t) + \varepsilon[u_1(\rho, \tau) + g_1(t)] \\
&v_0 = \delta^2\left[-\frac{1}{2}\rho^2\theta^{-1} + \delta C\exp(-\alpha_0\tau) + O(\delta^2)\right], \quad \tau = (1-\rho)/\delta \\
&u_0 = 2\theta + 2\delta C\theta_0^2\exp(-\alpha_0\tau) + O(\delta^2), \quad \theta_0 = \theta(1) \\
&\alpha_0^2 = \theta_0^2 f_{00}^{-1} = -2\theta_0^3[1 + O(\delta)], \quad g_0(t) = -u_0(1)K = -2\theta_0[1 + 2C\delta\theta_0 + O(\delta^2)]K \\
&K = \exp[-t\sqrt{v_0(1)}] = \exp[-\delta t a], \quad a = [-{}^1/_2\theta_0^{-1} + C\delta + \\
&\quad O(\delta^2)]^{1/2}, \quad h_2(t) = \frac{u_0(1)}{v_0(1)}K\left[u_0(1) - \theta(1) - \frac{1}{8}u_0(1)K\right] = \\
&\quad -4\delta^{-2}\theta_0^3(1 - {}^1/_4\exp(-\delta at))\exp(-\delta at)[1 + O(\delta)] \\
&v_1 = \theta_0\exp(-\alpha_0\tau)[1 + O(\delta)], \quad u_1 = 2\theta_0^3\delta^{-2}\exp(-\alpha_0\tau)\times \\
&\quad [1 + O(\delta)], \quad g_1(t) = -2\theta_0^3\delta^{-2}\exp(-a\delta t)\times[1 + \delta t(-2\theta_0)^{-1/2} + O(\delta)] \\
&\qquad\qquad C = 2^{-1.5}(-\theta_0)^{-3.5}[(2-\sigma)\theta_0 + \theta_1], \quad \theta_1 = -\left.\frac{d\theta}{d\rho}\right|_{\rho=1}
\end{aligned} \tag{4.5}$$

(there are misprints in the formula for $h_2(t)$ in /3/ that are eliminated here). Let us note that $v_{1k} = \omega_{1k} = 0$ for all k. Moreover, it can be established that the functions g_s°, h_{s+2} from (4.1) consist of components of the form $t^m\exp(-h\delta at)$, where m and n are integers $(0 \leqslant m \leqslant s+1,\ 1 \leqslant n \leqslant s+1)$. The coefficients between these components are expanded in a series of integer powers of δ, where the following estimates hold

$$\frac{d^\varkappa h_{s+2}^\circ}{dt^\varkappa} = O(1), \quad \frac{d^\varkappa g_s^\circ}{dt^\varkappa} = O(1), \quad \varkappa = 0, 1, 2 \tag{4.6}$$

Theorem 4.1. Let the conditions $\delta \ll 1$, $\varepsilon\delta^{-2} \ll 1$ and $(\varepsilon\delta^{-2})^{n+1}\varepsilon^{-4} \to 0$ $(n \gg 1)$ be satisfied simultaneously. Then a membrane solution of the problem (1.1), (1.2) exists for which the asymptotic formulas (4.5) are valid and the following estimates exist

$$\max|\delta^{-2}(v - v_{\varepsilon,\delta})| + \max|u - u_{\varepsilon,\delta}| < m_1(\varepsilon\delta^{-2})^2 \tag{4.7}$$

The theorem is proved analogously to the proof of Theorem 4.1 in /3/. The estimates (4.13) and (4.17) in /3/ are conserved, but in place of the estimate (4.9) we deduce for the same notation

$$\|\mathbf{P}(\mathbf{V}_k^*)\|_{L_\rho} < m_*(\varepsilon\delta^{-2})^{k+1} \quad (\delta \to 0,\ \varepsilon\delta^{-2} \to 0) \tag{4.8}$$

where m_* is a positive constant independent of δ and ε. For $(\varepsilon/\delta^2)^{n+1}\varepsilon^{-4} \to 0$ $(n \gg 1$, i.e., in the domain somewhat smaller than $\varepsilon/\delta^2 \ll 1)$ we obtain that all the conditions of Theorem 4.2 in /3/ are satisfied. From this theorem the existence of a membrane solution and the foundation of the asymptotic expansions (1.4) result. The estimates (4.5) are derived analogously to /13/ by using the trinagle inequalities.

Under the effect of internal pressure $(q < 0)$ the asymptotic expansions of the membrane solution and their foundation are carried out exactly the same as for $q > 0$. For the positive solution of the problem (1.3) for $\delta^2 = -q \to 0$ we have the asymptotic expansions (2.2)—(2.5) in which it is necessary to set $y_0 = 0$, to replace θ by $(-\theta)$ in (2.3) and θ_0 by $(-\theta_0)$ in (2.4) and (2.5). Limiting ourselves to just the principal terms of the asymptotic for the membrane solution for $|q| \to 0$, $\varepsilon/|q| \to 0$, we obtain the asymptotic formulas (4.5) but with u_0, g_0, h_2, v_1, u_1, g_1 replaced, respectively, by the following expressions:

$$\begin{aligned}
&u_0 = -2\delta\theta_0^2 C\exp(-\alpha_0\tau) + O(\delta^2), \quad \tau = (1-\rho)/\delta \\
&g_0(t) = \delta\exp(-\delta at)[2C\theta_0^2 + O(\delta)] \\
&h_2(t) = 2C\theta_0^3 a^{-2}\delta^{-1}\exp(-\delta at)[1 + O(\delta)] \\
&v_1(\rho, \tau) = \delta\,\alpha_0^{-1}[(2-\sigma)\theta_0 + \theta_1]\exp(-\alpha_0\tau)[1 + O(\delta)] \\
&u_1(\rho, \tau) = -2\theta^2\alpha_0^{-1}\delta^{-1}[(2-\sigma)\theta_0 + \theta_1]\exp(-\alpha_0\tau)[1 + O(\delta)]
\end{aligned} \tag{4.9}$$

5. Thin shallow shell in the shape of an elliptic paraboloid under uniform internal pressure. A secondary edge effect can occur for strictly convex elastic shells with fixed clamping of the edge. As an example we consider a shell initially in the shape of a thin elliptic paraboloid under uniform internal pressure. The appropriate equations are written in dimensionless variables in the form

$$\Delta^2 F + \frac{1}{2}[w, w] - [z, w] = 0, \quad \varepsilon^2\Delta^2 w - [w - z, F] + q = 0 \tag{5.1}$$
$$\Delta F = F_{xx} + F_{yy}, \quad [w, F] = w_{xx}F_{yy} + w_{yy}F_{xx} - 2w_{xy}F_{xy}$$
$$z = 1 - {}^1/_2\,(k_1x^2 + k_2y^2), \quad q > 0$$

Here k_1, k_2 are positive constants. For rigid clamping of the shell at the contour

$$\Gamma \equiv \{x = \sqrt{2/k_1}\cos\varphi,\ y = \sqrt{2/k_2}\sin\varphi,\ 0 \leqslant \varphi \leqslant 2\pi\}$$

we have the boundary conditions /15/:

$$w = w_\rho = 0, \quad \Gamma_2 F \equiv F_{\rho\rho} - \sigma F_{ss} + \varkappa\sigma F_\rho = 0 \tag{5.2}$$
$$\Gamma_3 F \equiv F_{\rho\rho\rho} + (2 + \sigma)F_{\rho ss} - 3\varkappa F_{ss} + (2 + \sigma)\varkappa_s F_s - \varkappa^2 \times (1 - \sigma)F_\rho = 0$$

($\rho, s, \varkappa$ are, respectively, the internal normal, the arc length, and the curvature of Γ). The "degenerate" problem about the equilibrium of an absolutely flexible shell when $\varepsilon = 0$ is written in the form of a system of equations with nonlinear boundary (edge) conditions (*)

$$\Delta^2 F_0 + {}^1/_2[w_0, w_0] - [z, w_0] = 0, \quad [w_0 - z, F_0] = q \tag{5.3}$$
$$w_0|_\Gamma = 0, \quad \Gamma_2 F_0 = 0, \quad \Gamma_3 F_0 = \varkappa\,({}^1/_2 w_{0\rho} - z_\rho)w_{0\rho}|_\Gamma$$

The so-called positive solutions satisfying the following conditions

$$F_{0xx} \geqslant 0, \quad F_{0yy} \geqslant 0, \quad F_{0xx}F_{0yy} - F_{0xy}{}^2 \geqslant 0 \tag{5.4}$$

everywhere in $D + \Gamma$ have meaning in mechanics for the boundary value problem (5.3) /5,16,17/, where D is the domain the shell planform occupies.

The positive solutions introduced in Sect.1 for the axisymmetric equilibria satisfy the conditions (5.4) only upon compliance with the additional inequality $dv_0/dr \geqslant 0$ for $r \in [0, 1]$.

Then as $q \to 0$ the following asymptotic representations are constructed for the positive solution of the problem (5.3)

$$F_0{}^q \approx q\,[f\,(x, y) + q\psi_1\,(\tau, \varphi)], \quad w_0{}^q \approx q\,[(\sigma^2 - 1)\,K + \beta_1\,(\tau, \varphi)] \tag{5.5}$$
$$f = {}^1/_2 K\,[(k_2 + \sigma k_1)x^2 + (k_1 + \sigma k_2)y^2], \quad \tau = \rho/\sqrt{q}$$
$$K = (k_1{}^2 + k_2{}^2 + 2\sigma k_1k_2)^{-1}, \quad \beta_1 = -T_n\,(\varkappa c)^{-1}\psi_1 =$$
$$(\varkappa c)^{-1}\exp\,(-\alpha\tau)\Gamma_2 f, \quad c = z_\rho|_\Gamma, \quad \alpha = \varkappa c T_n^{-1/2} > 0$$
$$\varkappa c = k_1k_2b_0 > 0, \quad T_n = Kb_0\,(k_2{}^2\sin^2\varphi + k_1{}^2\cos^2\varphi + \sigma k_1k_2)$$
$$b_0 = (k_1\cos^2\varphi + k_2\sin^2\varphi)^{-1}, \quad \Gamma_2 f = (1 - \sigma^2)b_0k_1k_2K$$

Theorem 5.1. Let $0.5 \leqslant k_1k_2^{-1} \leqslant 2$ and $q \to 0$. Then there exists a q_1 such that for $0 < q < q_1$ the problem (5.3) has a unique positive solution (F_*, w_*) for which the asymptotic representations (5.5) are valid, and the following estimates hold for $j = 0, 1, 2$

$$\max\,[\,|D^j\,(w_* - w_0{}^q)\,| + |\,D^j\,(F_* - F_0{}^q)\,|\,] \leqslant m_1 q^{(3-j)/2}$$

where m_1 is a certain positive constant independent of q and D^j is the derivative of order j.

The proof of the theorem is omitted and will be presented somewhere else. Compliance with the inequalities (5.4) for $F_0{}^q$ in the conditions of the theorem is established by direct computations.

Using the presence of the second small parameter ε^2 in (5.1) and applying the boundary-layer method according to the scheme of /17/, we obtain that the problem (5.1), (5.2) has a

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*) See Srubshchik, L.S., Edge effect in the flexure of absolutely flexible shells. Dep. VINITI, No.4265-79, 14 December 1979, Rostov-on-Don.

solution in the neighborhood of the positive solution (5.5) as the parameters q and εq^{-1} tend simultaneously to zero, for which the following simple asymptotics formulas are valid:

$$F \approx F_0^q(x, y) + \varepsilon^3 h_2(t, \varphi), \quad w \approx w_0^q(x, y) + \varepsilon g_0(t, \varphi) \qquad (5.6)$$

$$g_0 = a_1 m^{-1} \exp(-\rho m/\varepsilon), \quad \partial^2 h_2/\partial t^2 = \varkappa a_1 m^{-1} [(a_1 - c)\exp(-\rho m/\varepsilon) - {}^1/_4 a_1 \exp(-2\rho m/\varepsilon)]$$

$$m = \sqrt{N}, \quad a_1 = \frac{\partial w_0}{\partial \rho}\bigg|_{\rho=0}, \quad c = \frac{\partial z}{\partial \rho}\bigg|_{\rho=0}, \quad N = [F_{0ss} - \varkappa F_{0\rho}]_\Gamma > 0, \quad t = \rho/\varepsilon$$

The solution (5.6) corresponds to equilibrium with just tensile forces, and the phenomenon of the secondary edge effect holds for it.

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Translated by M.D.F.